

LEFT-ORDERABLE FUNDAMENTAL GROUP AND DEHN SURGERY ON TWIST KNOTS

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ABSTRACT. For any hyperbolic twist knot in the 3-sphere, we show that the resulting manifold by r -surgery on the knot has left-orderable fundamental group if the slope r satisfies the inequality $0 \leq r \leq 4$.

1. INTRODUCTION

A non-trivial group G is said to be *left-orderable* if it admits a strict total ordering which is invariant under left-multiplication. Thus, if $g < h$ then $fg < fh$ for any $f, g, h \in G$. Many groups, which arise in topology such as orientable surface groups, knot groups, braid groups, are known to be left-orderable. In 3-manifold topology, it is natural to ask which 3-manifolds have left-orderable fundamental groups. Toward this direction, there is very recent evidence of connections between Heegaard-Floer homology and left-orderability of fundamental groups. More precisely, Boyer, Gordon and Watson [3] conjecture that an irreducible rational homology 3-sphere is an L -space if and only if its fundamental group is not left-orderable. An L -space is a rational homology 3-sphere Y whose Heegaard-Floer homology group $\widehat{HF}(Y)$ has rank equal to $|H_1(Y; \mathbb{Z})|$ ([18]). They confirmed the conjecture for several classes of 3-manifolds including Seifert fibered manifolds, Sol-manifolds. Also, they showed that if $-4 < r < 4$ then r -surgery on the figure-eight knot yields a 3-manifold whose fundamental group is left-orderable. Later, Clay, Lidman and Watson [6] added the same conclusion for $r = \pm 4$. Since the figure-eight knot cannot yield L -spaces by non-trivial Dehn surgery ([18]), these give supporting evidences of the conjecture.

The purpose of this paper is to push forward them to all hyperbolic twist knots. Any non-trivial twist knot except the trefoil is hyperbolic, and does not admit non-trivial Dehn surgery yielding L -spaces ([18]). Hence the following result gives a further supporting evidence of the conjecture mentioned above.

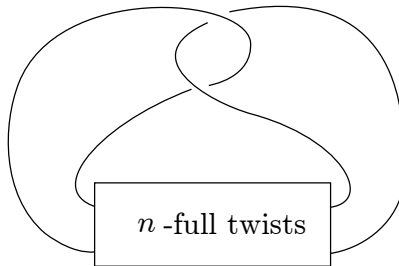
Theorem 1.1. *Let K be a hyperbolic twist knot in the 3-sphere S^3 as illustrated in Figure 1. If $0 \leq r \leq 4$, then r -surgery on K yields a manifold whose fundamental group is left-orderable.*

As seen in Figure 1, the clasp is left-handed. The range of the slope in the conclusion of Theorem 1.1 depends on this convention. If the clasp is right-handed, the range would be $[-4, 0]$.

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FIGURE 1. The n -twist knot

For the right-handed trefoil, if $r \geq 1$, then r -surgery yields an L -space ([13]), and its fundamental group is not left-orderable by [3]. Otherwise, r -surgery yields a manifold with left-orderable fundamental group ([8]).

In [11], we showed the same conclusion as Theorem 1.1 for the knot 5_2 , which is the (-2) -twist knot. We will greatly generalize the argument in [11] to handle all hyperbolic twist knots. Our argument works even for the figure-eight knot, and it is much simpler than that in [3], which involves character varieties. We remark that the fact that the figure-eight knot is amphicheiral makes possible to widen the range of slope to $-4 \leq r \leq 4$.

2. KNOT GROUP AND REPRESENTATIONS

Let K_n be the n -twist knot with diagram illustrated in Figure 1. Our convention is that the twists are right-handed if $n > 0$ and left-handed if $n < 0$. Thus K_1 is the figure-eight knot and K_{-1} is the right-handed trefoil. If $n \neq 0, -1$, then K_n is hyperbolic, and if $|n| > 1$, then K_n is not fibered. Throughout the paper, we assume that $n \neq 0, -1$. Thus non-trivial Dehn surgery on K_n never yields an L -space ([18]).

It is easy to see from the diagram that K_n bounds a once-punctured Klein bottle whose boundary slope is 4. (For example, consider a checkerboard coloring of the diagram. Then the bounded surface gives such a once-punctured Klein bottle.) Thus, 4-surgery on K_n yields a non-hyperbolic manifold, which is a toroidal manifold. In [22], we showed that the resulting toroidal manifold by 4-surgery on K_n admits a left-ordering on its fundamental group. Also, 1, 2 and 3-surgeries on K_n are known to yield small Seifert fibered manifolds ([5]), and the resulting manifolds have left-orderable fundamental groups by [3]. However, we do not need the latter fact.

Let $G = \pi_1(S^3 - K_n)$ be the knot group of K_n .

Lemma 2.1. *The knot group G admits a presentation*

$$G = \langle x, y \mid w^n x = y w^n \rangle,$$

where x and y are meridians and $w = xy^{-1}x^{-1}y$.

Furthermore, the longitude \mathcal{L} is given as $\mathcal{L} = w_*^n w^n$, where $w_* = yx^{-1}y^{-1}x$ is obtained from w by reversing the order of letters.

This is slightly different from that in [14, Proposition 1], but they are isomorphic.

Proof. We use the surgery diagram of K_n as illustrated in Figure 2, where 1-surgery and $-1/n$ -surgery are performed along the second and third components, as indicated by numbers, respectively.

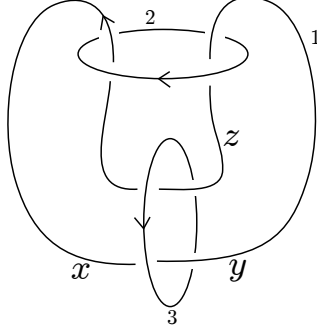


FIGURE 2. A surgery diagram of K_n

Let μ_i and λ_i be the meridian and longitude of the i -th component. Then $y = \mu_3^{-1}x\mu_3$, $z = \mu_2^{-1}y\mu_2$, $\lambda_2 = x^{-1}y$ and $\lambda_3 = yz^{-1}$. By 1-surgery on the second component, the relation $\lambda_2\mu_2 = 1$ arises. Similarly, $-1/n$ -surgery on the third component implies $\lambda_3^n\mu_3^{-1} = 1$.

Hence, $\mu_3^{-1} = \lambda_3^{-n} = (zy^{-1})^n = (x^{-1}yxy^{-1})^n$. Let $w = xy^{-1}x^{-1}y$. Then $x^{-1}yxy^{-1} = x^{-1}ywy^{-1}x$, so $\mu_3^{-1} = x^{-1}yw^ny^{-1}x$. By substituting this to the remaining relation $y = \mu_3^{-1}x\mu_3$, we obtain $w^nx = yw^n$.

Finally, the longitude \mathcal{L} is given as $\mu_3\mu_2\mu_3^{-1}\mu_2^{-1} = x^{-1}yw^{-n}y^{-1}xw^n$. Since $x^{-1}yw^{-n}y^{-1}x = (yx^{-1}y^{-1}x)^n$, we have $\mathcal{L} = w_*^nw^n$, where $w_* = yx^{-1}y^{-1}x$. \square

Let $s > 0$ and $t > 1$ be real numbers. Let $\rho_s : G \rightarrow SL_2(\mathbb{R})$ be the representation defined by the correspondence

$$(2.1) \quad \rho_s(x) = \begin{pmatrix} \sqrt{t} & 0 \\ 0 & \frac{1}{\sqrt{t}} \end{pmatrix}, \quad \rho_s(y) = \begin{pmatrix} \frac{t-s-1}{\sqrt{t}-\frac{1}{\sqrt{t}}} & \frac{s}{(\sqrt{t}-\frac{1}{\sqrt{t}})^2} - 1 \\ -s & \frac{s+1-\frac{1}{t}}{\sqrt{t}-\frac{1}{\sqrt{t}}} \end{pmatrix}.$$

$$\text{For } P = \begin{pmatrix} t-1 & 1 \\ 0 & \sqrt{t}-\frac{1}{\sqrt{t}} \end{pmatrix},$$

$$P^{-1}\rho_s(x)P = \begin{pmatrix} \sqrt{t} & \frac{1}{\sqrt{t}} \\ 0 & \frac{1}{\sqrt{t}} \end{pmatrix}, \quad P^{-1}\rho_s(y)P = \begin{pmatrix} \sqrt{t} & 0 \\ -s\sqrt{t} & \frac{1}{\sqrt{t}} \end{pmatrix}.$$

Hence, (2.1) gives a (non-abelian) representation if s and t satisfy Riley's equation $z_{1,1} + (1-t)z_{1,2} = 0$, where $z_{i,j}$ is the (i,j) -entry of the matrix $P^{-1}\rho_s(w^n)P$ ([20]). See also [9]. Then $\phi_n(s,t) = z_{1,1} + (1-t)z_{1,2}$ is called the Riley polynomial of K_n .

Since s and t are limited to be positive real numbers in our setting, it is not obvious that there exist solutions for Riley's equation $\phi_n(s,t) = 0$. However, this will be verified in Proposition 3.2. We temporarily assume that s and t are chosen so that $\phi_n(s,t) = 0$.

From (2.1), we have

$$(2.2) \quad W = \rho_s(w) = \begin{pmatrix} 1 + s - st + \frac{s^2 t}{t-1} & \frac{t-1+st}{\sqrt{t}} \frac{(\sqrt{t}-\frac{1}{\sqrt{t}})^2-s}{(\sqrt{t}-\frac{1}{\sqrt{t}})^2} \\ \frac{s(1+s-t)}{\sqrt{t}} & 1 + s - \frac{s^2}{t-1} - \frac{s}{t} \end{pmatrix}.$$

Let

$$(2.3) \quad \lambda_{\pm} = \frac{1}{2} \left\{ s^2 - \left(t + \frac{1}{t} - 2 \right) s + 2 \pm \sqrt{\left(s^2 - \left(t + \frac{1}{t} - 2 \right) s + 2 \right)^2 - 4} \right\} \in \mathbb{C}.$$

These are eigenvalues of W , and so $\lambda_+ + \lambda_- = \text{tr}(W) = s^2 - (t + 1/t - 2)s + 2$ and $\lambda_+ \lambda_- = 1$. In Proposition 3.2, we will see that $s + 2 < t + 1/t < s + 2 + 4/s$. This implies

$$-2 < s^2 - \left(t + \frac{1}{t} - 2 \right) s + 2 < 2,$$

and so $\lambda_{\pm} = e^{\pm i\theta}$ for some $\theta \in (0, \pi)$. In particular, we remark that $\lambda_+ \neq \lambda_-$.

Proposition 2.2. *The Riley polynomial $\phi_n(s, t)$ of K_n is given by*

$$(2.4) \quad \frac{\lambda_+^{n+1} - \lambda_-^{n+1}}{\lambda_+ - \lambda_-} - \left(t + \frac{1}{t} - 1 - s \right) \frac{\lambda_+^n - \lambda_-^n}{\lambda_+ - \lambda_-}.$$

Proof. The Riley polynomial is explicitly calculated in [16, Proposition 3.1]. Our knot K_n corresponds to the mirror image of $J(2, -n)$ in [16]. This gives the conclusion. \square

By using Lemma 4.4, it is not hard to check directly that $\rho_s(w^n x) = \rho_s(yw^n)$ holds if and only if s and t make the polynomial (2.4) equal to zero.

Set $T = t + 1/t$ and $\tau_m = (\lambda_+^m - \lambda_-^m)/(\lambda_+ - \lambda_-)$ for convenience. Then the Riley polynomial of K_n is expressed simply as $\phi_n(s, T) = \tau_{n+1} - (T - 1 - s)\tau_n$. Since τ_m is symmetric in λ_+ and λ_- , it can be expressed as a polynomial of $\lambda_+ + \lambda_-$, which is $s^2 - (T - 2)s + 2$. Also, it is easy to see that a recursive relation

$$(2.5) \quad \tau_{m+1} - (\lambda_+ + \lambda_-)\tau_m + \tau_{m-1} = 0$$

and $\tau_{-m} = -\tau_m$ hold for any integer m .

Example 2.3. Clearly, $\tau_0 = 0$ and $\tau_1 = 1$. Thus we have $\tau_2 = s^2 - (T - 2)s + 2$ and $\tau_3 = (s^2 - (T - 2)s + 2)^2 - 1$. From these, the figure-eight knot has the Riley polynomial

$$\phi_1(s, T) = \tau_2 - (T - 1 - s)\tau_1 = -(s + 1)T + s^2 + 3s + 3.$$

Similarly, the 2-twist knot, 6_1 in the knot table, has the Riley polynomial

$$\begin{aligned} \phi_2(s, T) &= \tau_3 - (T - 1 - s)\tau_2 \\ &= (s^2 + s)T^2 - (2s^3 + 6s^2 + 7s + 2)T + s^4 + 5s^3 + 11s^2 + 12s + 5. \end{aligned}$$

From the recursive relation (2.5), we see that the Riley polynomial $\phi_n(s, T)$ has degree $|n|$ in T . Thus we cannot solve the equation $\phi_n(s, T) = 0$ for T , in general.

3. RILEY POLYNOMIALS

In this section, we show that Riley's equation $\phi_n(s, T) = 0$ has a pair of solutions (s, T) such as $s + 2 < T < s + 2 + 4/s$ for any $s > 0$. In fact, we can choose T satisfying $s + 2 + c/s < T < s + 2 + 4/s$ where c is a constant depending only n , unless $n = 1$.

Let m be a positive integer. For $z = e^{i\theta}$ ($0 \leq \theta \leq \pi$), set $\mathcal{T}_m(z) = z^{m-1} + z^{m-3} + \dots + z^{3-m} + z^{1-m}$. If $z \neq \pm 1$, then $\mathcal{T}_m(z) = (z^m - z^{-m})/(z - z^{-1})$. Define $\mathcal{T}_0 = 0$ and $\mathcal{T}_{-m}(z) = -\mathcal{T}_m(z)$. Since $\mathcal{T}_m(z)$ is symmetric for z and z^{-1} , it can be expanded as a polynomial of $z + z^{-1}$. Furthermore, the recursive relation

$$\mathcal{T}_{m+1}(z) - (z + z^{-1})\mathcal{T}_m(z) + \mathcal{T}_{m-1}(z) = 0$$

holds. Also, $\mathcal{T}_m(1) = m$ and $\mathcal{T}_m(-1) = (-1)^{m-1}m$ for any integer m .

Lemma 3.1. (1) Let $m \geq 1$. Then, $\mathcal{T}_m(e^{\frac{\pi}{2m+1}i}) = \mathcal{T}_{m+1}(e^{\frac{\pi}{2m+1}i})$, and this value is positive.

(2) Let $m \geq 2$. Then, $\mathcal{T}_m(e^{\frac{3\pi}{2m+1}i}) = \mathcal{T}_{m+1}(e^{\frac{3\pi}{2m+1}i})$, and this value is negative.

Proof. (1) Let $z = e^{\frac{\pi}{2m+1}i}$. Then the fact that $z^{2m+1} = -1$ immediately implies $\mathcal{T}_m(z) = \mathcal{T}_{m+1}(z)$. A direct calculation shows

$$\mathcal{T}_m(z) = \frac{\sin \frac{m\pi}{2m+1}}{\sin \frac{\pi}{2m+1}} > 0.$$

(2) Similarly, set $z = e^{\frac{3\pi}{2m+1}i}$. Then $z^{2m+1} = -1$ holds again. Hence we have $\mathcal{T}_m(z) = \mathcal{T}_{m+1}(z)$, and

$$\mathcal{T}_m(z) = \frac{\sin \frac{3m\pi}{2m+1}}{\sin \frac{3\pi}{2m+1}} < 0.$$

□

Now, fix an $s > 0$. We introduce a function $\Phi_n : [s + 2, s + 2 + 4/s] \rightarrow \mathbb{R}$ by

$$(3.1) \quad \Phi_n(T) = \mathcal{T}_{n+1}(z) - (T - 1 - s)\mathcal{T}_n(z),$$

where

$$z = \frac{1}{2} \left\{ s^2 - (T - 2)s + 2 + i\sqrt{4 - (s^2 - (T - 2)s + 2)^2} \right\}.$$

Since $s + 2 \leq T \leq s + 2 + 4/s$, we have $-2 \leq s^2 - (T - 2)s + 2 \leq 2$. Thus $z = e^{i\theta}$ for $\theta \in [0, \pi]$. We will seek a solution T for $\Phi_n(T) = 0$ satisfying $s + 2 < T < s + 2 + 4/s$, because it gives a pair of solutions (s, T) for Riley's equation $\phi_n(s, T) = 0$.

Proposition 3.2. *Riley's equation $\phi_n(s, T) = 0$ has a real solution T satisfying $s + 2 < T < s + 2 + 4/s$ for any $s > 0$. Moreover, if $n \neq 1$, then T can be chosen so that $s + 2 + c/s < T < s + 2 + 4/s$, where c is a constant depending only on n . In particular, $\phi_n(s, t) = 0$ has a solution $t > 1$ for any $s > 0$.*

Proof. Suppose $n > 1$. By Lemma 3.1,

$$\mathcal{T}_{n+1}(e^{\frac{\pi}{2n+1}i}) = \mathcal{T}_n(e^{\frac{\pi}{2n+1}i}), \quad \mathcal{T}_{n+1}(e^{\frac{3\pi}{2n+1}i}) = \mathcal{T}_n(e^{\frac{3\pi}{2n+1}i}).$$

Let $c = 2 - 2 \cos \frac{\pi}{2n+1}$ and $c' = 2 - 2 \cos \frac{3\pi}{2n+1}$. Then $c, c' \in (0, 4)$ and $c < c'$.

Also,

$$\begin{aligned}
\Phi_n(s+2+\frac{c}{s}) &= \mathcal{T}_{n+1}(e^{\frac{\pi}{2n+1}i}) - \left(1+\frac{c}{s}\right) \mathcal{T}_n(e^{\frac{\pi}{2n+1}i}) \\
&= -\frac{c}{s} \cdot \mathcal{T}_n(e^{\frac{\pi}{2n+1}i}), \\
\Phi_n(s+2+\frac{c'}{s}) &= \mathcal{T}_{n+1}(e^{\frac{3\pi}{2n+1}i}) - \left(1+\frac{c'}{s}\right) \mathcal{T}_n(e^{\frac{3\pi}{2n+1}i}) \\
&= -\frac{c'}{s} \cdot \mathcal{T}_n(e^{\frac{3\pi}{2n+1}i}).
\end{aligned}$$

By Lemma 3.1, these values have distinct signs. We remark that $\Phi_n(T)$ is a polynomial function of T , so it is continuous. Thus if $n > 1$, we have a solution T for $\Phi_n(T) = 0$, satisfying $s+2+c/s < T < s+2+c'/s$, from the Intermediate-Value Theorem. Since $T > 2$, $t+1/t = T$ has a real solution for t . If we choose $t = (T + \sqrt{T^2 - 4})/2$, then $t > 1$.

When $n = 1$, the Riley polynomial is $\phi_1(s, T) = -(s+1)T + s^2 + 3s + 3$ as shown in Example 2.3. Hence the equation $\phi_1(s, T) = 0$ has the unique solution $T = (s^2 + 3s + 3)/(s+1) = s+2+1/(s+1)$ for a given s . This satisfies $s+2 < T < s+2+1/s$.

Suppose $n < 0$. (Recall that we assume $n \neq -1$.) Set $l = |n| \geq 2$. If $l > 2$, then set $d = 2 - 2\cos\frac{\pi}{2l-1}$ and $d' = 2 - 2\cos\frac{3\pi}{2l-1}$. Then $d, d' \in (0, 4)$ and $d < d'$. As before,

$$\mathcal{T}_{l-1}(e^{\frac{\pi}{2l-1}i}) = \mathcal{T}_l(e^{\frac{\pi}{2l-1}i}), \quad \mathcal{T}_{l-1}(e^{\frac{3\pi}{2l-1}i}) = \mathcal{T}_l(e^{\frac{3\pi}{2l-1}i}).$$

by Lemma 3.1. Thus

$$\begin{aligned}
\Phi_n(s+2+\frac{d}{s}) &= -\left(\mathcal{T}_{l-1}(e^{\frac{\pi}{2l-1}i}) - \left(1+\frac{d}{s}\right) \mathcal{T}_l(e^{\frac{\pi}{2l-1}i})\right) \\
&= \frac{d}{s} \cdot \mathcal{T}_l(e^{\frac{\pi}{2l-1}i}), \\
\Phi_n(s+2+\frac{d'}{s}) &= \frac{d'}{s} \cdot \mathcal{T}_l(e^{\frac{3\pi}{2l-1}i}).
\end{aligned}$$

Since these values have distinct signs, we have a solution T with $s+2+d/s < T < s+2+d'/s$, if $l > 2$, as before.

When $l = 2$, we have

$$\Phi_{-2}(s+2+\frac{1}{s}) = \frac{1}{s} > 0, \quad \Phi_{-2}(s+2+\frac{2}{s}) = -1 < 0.$$

Hence there exists a solution T with $s+2+1/s < T < s+2+2/s$. \square

4. LONGITUDES

Recall that $\rho_s : G \rightarrow SL_2(\mathbb{R})$ is the representation defined by (2.1). Two real parameters s and t are chosen so that $\phi_n(s, t) = 0$. In this section, we examine the image of the longitude \mathcal{L} of G under ρ_s . Throughout the section, let

$$\rho_s(w) = \begin{pmatrix} w_{1,1} & w_{1,2} \\ w_{2,1} & w_{2,2} \end{pmatrix}, \quad \rho_s(w^n) = \begin{pmatrix} u_{1,1} & u_{1,2} \\ u_{2,1} & u_{2,2} \end{pmatrix}$$

$$\text{and } \sigma = \frac{s(\sqrt{t}-\frac{1}{\sqrt{t}})^2}{(\sqrt{t}-\frac{1}{\sqrt{t}})^2-s}.$$

Lemma 4.1. For w_* , we have $\rho_s(w_*) = \begin{pmatrix} u_{1,1} & \frac{u_{2,1}}{\sigma} \\ u_{1,2}\sigma & u_{2,2} \end{pmatrix}$.

Proof. By a direct calculation,

$$\begin{aligned} \rho_s(xy^{-1}) &= \begin{pmatrix} \frac{t-1+st}{t-1} & \frac{s\sqrt{t}}{\frac{\sigma}{\sqrt{t}}} \\ \frac{s}{\sqrt{t}} & \frac{t-1-s}{t-1} \end{pmatrix}, & \rho_s(y^{-1}x) &= \begin{pmatrix} \frac{t-1+st}{t-1} & \frac{s}{\sqrt{t}\sigma} \\ s\sqrt{t} & \frac{t-1-s}{t-1} \end{pmatrix}, \\ \rho_s(x^{-1}y) &= \begin{pmatrix} \frac{t-1-s}{t-1} & -\frac{s}{\sqrt{t}\sigma} \\ -s\sqrt{t} & \frac{t-1+st}{t-1} \end{pmatrix}, & \rho_s(yx^{-1}) &= \begin{pmatrix} \frac{t-1-s}{t-1} & -\frac{s\sqrt{t}}{\sigma} \\ -\frac{s}{\sqrt{t}} & \frac{t-1+st}{t-1} \end{pmatrix}. \end{aligned}$$

Thus we see that the $(1, 2)$ -entry of $\rho_s(y^{-1}x)$ is the $(2, 1)$ -entry of $\rho_s(xy^{-1})$ divided by σ , the $(2, 1)$ -entry of $\rho_s(y^{-1}x)$ is the $(1, 2)$ -entry of $\rho_s(xy^{-1})$ multiplied by σ , and the others of $\rho_s(y^{-1}x)$ coincide with those of $\rho_s(xy^{-1})$. The same relation between entries holds for $\rho_s(x^{-1}y)$ and $\rho_s(yx^{-1})$.

In general, such a relation is preserved under the matrix multiplication;

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{pmatrix},$$

$$\begin{pmatrix} p & \frac{r}{\sigma} \\ q\sigma & s \end{pmatrix} \begin{pmatrix} a & \frac{c}{\sigma} \\ b\sigma & d \end{pmatrix} = \begin{pmatrix} ap+br & \frac{cp+dr}{\sigma} \\ (aq+bs)\sigma & cq+ds \end{pmatrix}.$$

Thus we can confirm that the same relation holds for $\rho_s(w_*)$ and $\rho_s(w^n)$. \square

Proposition 4.2. For the longitude \mathcal{L} of G , the matrix $\rho_s(\mathcal{L})$ is diagonal, and the $(1, 1)$ -entry of $\rho_s(\mathcal{L})$ is a positive real number.

Proof. The first assertion follows from the facts that for a meridian x , $\rho_s(x)$ is diagonal but $\rho_s(x) \neq \pm I$ and that x and \mathcal{L} commute.

Since $\mathcal{L} = w_*^n w^n$ by Lemma 2.1, Lemma 4.1 implies that

$$\begin{aligned} \rho_s(\mathcal{L}) = \rho_s(w_*)\rho_s(w^n) &= \begin{pmatrix} u_{1,1} & \frac{u_{2,1}}{\sigma} \\ u_{1,2}\sigma & u_{2,2} \end{pmatrix} \begin{pmatrix} u_{1,1} & u_{1,2} \\ u_{2,1} & u_{2,2} \end{pmatrix} \\ &= \begin{pmatrix} u_{1,1}^2 + \frac{u_{2,1}^2}{\sigma} & u_{1,1}u_{1,2} + \frac{u_{2,1}u_{2,2}}{\sigma} \\ u_{1,1}u_{1,2}\sigma + u_{2,1}u_{2,2} & u_{1,2}^2\sigma + u_{2,2}^2 \end{pmatrix}. \end{aligned}$$

Since $\det \rho_s(w^n) = 1$, at least one of $u_{1,1}$ and $u_{2,1}$ is non-zero. Hence the $(1, 1)$ -entry is $u_{1,1}^2 + u_{2,1}^2/\sigma$, which is positive, because $s > 0$ and $(\sqrt{t} - 1/\sqrt{t})^2 - s = T - s - 2 > 0$ from Proposition 3.2. \square

Remark 4.3. Since $\rho_s(\mathcal{L})$ is diagonal, we also obtain an equation $u_{1,1}u_{1,2}\sigma + u_{2,1}u_{2,2} = 0$. This will be used in the proof of Lemma 4.5.

To diagonalize $W = \rho_s(w)$, let $Q = \begin{pmatrix} w_{1,2} & w_{1,2} \\ \lambda_+ - w_{1,1} & \lambda_- - w_{1,1} \end{pmatrix}$. From (2.2), $w_{1,2} = (t - 1 + st)s/(\sigma\sqrt{t})$. Since $s > 0, t > 1, \sigma > 0$, we have $w_{1,2} \neq 0$. Also, $\det Q = -w_{1,2}(\lambda_+ - \lambda_-)$. Then a direct calculation shows $Q^{-1}WQ = \begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}$.

Lemma 4.4. The entries of W^n are given as follows.

$$\begin{aligned} u_{1,1} &= w_{1,1}\tau_n - \tau_{n-1}, & u_{1,2} &= w_{1,2}\tau_n, \\ u_{2,1} &= w_{2,1}\tau_n, & u_{2,2} &= \tau_{n+1} - w_{1,1}\tau_n. \end{aligned}$$

Proof. This easily follows from $W^n = Q \begin{pmatrix} \lambda_+^n & 0 \\ 0 & \lambda_-^n \end{pmatrix} Q^{-1}$.

For example,

$$\begin{aligned} u_{2,1} &= \frac{1}{\det Q} \left(\lambda_+^n (\lambda_+ - w_{1,1}) (\lambda_- - w_{1,1}) + \lambda_-^n (\lambda_- - w_{1,1}) (w_{1,1} - \lambda_+) \right) \\ &= -\frac{\tau_n}{w_{1,2}} (\lambda_+ - w_{1,1}) (\lambda_- - w_{1,1}) \\ &= -\frac{\tau_n}{w_{1,2}} (1 - \operatorname{tr}(W) w_{1,1} + w_{1,1}^2). \end{aligned}$$

Since $\operatorname{tr}(W) = w_{1,1} + w_{2,2}$, we have $1 - \operatorname{tr}(W) w_{1,1} + w_{1,1}^2 = 1 - w_{1,1} w_{2,2} = -w_{1,2} w_{2,1}$. Thus $u_{2,1} = w_{2,1} \tau_n$.

We omit the others. \square

Lemma 4.5. *Let B_s be the $(1, 1)$ -entry of the matrix $\rho_s(\mathcal{L})$. Then $B_s = -w_{2,1}/(w_{1,2}\sigma)$.*

Proof. As noted in Remark 4.3, $u_{1,1}u_{1,2}\sigma + u_{2,1}u_{2,2} = 0$. Since $\det W^n = u_{1,1}u_{2,2} - u_{1,2}u_{2,1} = 1$, we have

$$\begin{aligned} u_{1,2}B_s &= u_{1,1}^2 u_{1,2} + \frac{u_{1,2}u_{2,1}^2}{\sigma} \\ &= u_{1,1}^2 u_{1,2} + \frac{u_{2,1}}{\sigma} (u_{1,1}u_{2,2} - 1) \\ &= u_{1,1}^2 u_{1,2} + \frac{u_{1,1}}{\sigma} (-u_{1,1}u_{1,2}\sigma) - \frac{u_{2,1}}{\sigma} \\ &= -\frac{u_{2,1}}{\sigma}. \end{aligned}$$

By Lemma 4.4, $u_{1,2} = w_{1,2}\tau_n$. As remarked above Lemma 4.4, $w_{1,2} \neq 0$. If $u_{1,2} = 0$, then $\tau_n = 0$. But this implies $\tau_{n+1} = 0$, because $\phi_n(s, t) = \tau_{n+1} - (t + 1/t - 1 - s)\tau_n = 0$. From the recursive relation, this implies $\tau_m = 0$ for all m . But this is absurd, because $\tau_1 = 1$. Hence $u_{1,2} \neq 0$, so $B_s = -u_{2,1}/(u_{1,2}\sigma)$. From Lemma 4.4 again, $u_{1,2} = w_{1,2}\tau_n$ and $u_{2,1} = w_{2,1}\tau_n$. Thus $B_s = -w_{2,1}/(w_{1,2}\sigma)$. \square

5. LIMITS

Let $r = p/q$ be a rational number, and let $M_n(r)$ denote the resulting manifold by r -filling on the knot exterior M_n of K_n . In other words, $M_n(r)$ is obtained by attaching a solid torus V to M_n along their boundaries so that the loop $x^p \mathcal{L}^q$ bounds a meridian disk of V , where x and \mathcal{L} are a meridian and longitude of K_n .

Our representation $\rho_s : G \rightarrow SL_2(\mathbb{R})$ induces a homomorphism $\pi_1(M_n(r)) \rightarrow SL_2(\mathbb{R})$ if and only if $\rho_s(x)^p \rho_s(\mathcal{L})^q = I$. Since both of $\rho_s(x)$ and $\rho_s(\mathcal{L})$ are diagonal (see (2.1) and Proposition 4.2), this is equivalent to the equation

$$(5.1) \quad A_s^p B_s^q = 1,$$

where A_s and B_s are the $(1, 1)$ -entries of $\rho_s(x)$ and $\rho_s(\mathcal{L})$, respectively. We remark that $A_s = \sqrt{t}$ (> 1) is a positive real number, so is B_s by Proposition 4.2. Hence the equation (5.1) is furthermore equivalent to the equation

$$(5.2) \quad -\frac{\log B_s}{\log A_s} = \frac{p}{q}.$$

Let $g : (0, \infty) \rightarrow \mathbb{R}$ be a function defined by

$$g(s) = -\frac{\log B_s}{\log A_s}.$$

We will examine the image of g .

Lemma 5.1. (1) If $|n| > 1$, then $\lim_{s \rightarrow +0} t = \infty$. If $n = 1$, then $\lim_{s \rightarrow +0} t =$

$$(3 + \sqrt{5})/2.$$

$$(2) \lim_{s \rightarrow \infty} t = \infty.$$

$$(3) \lim_{s \rightarrow \infty} (t - s) = 2.$$

$$(4) \lim_{s \rightarrow \infty} \frac{s}{t} = 1.$$

Proof. (1) If $n = 1$, then the equation $\phi_1(s, T) = 0$ has the unique solution $T = (s^2 + 3s + 3)/(s + 1)$ for a given $s > 0$, so $\lim_{s \rightarrow +0} T = 3$. Since $t = (T + \sqrt{T^2 - 4})/2$, we have $\lim_{s \rightarrow +0} t = (3 + \sqrt{5})/2$.

Assume $|n| > 1$. From Proposition 3.2, we have $s + 2 + c/s < T$, where c is a positive constant. Hence $\lim_{s \rightarrow +0} T = \lim_{s \rightarrow +0} t = \infty$.

(2) As $T > s + 2$, $\lim_{s \rightarrow \infty} T = \lim_{s \rightarrow \infty} t = \infty$.

(3) Since $s + 2 < t + 1/t < s + 2 + 4/s$, (2) implies $\lim_{s \rightarrow \infty} (t - s) = 2$.

(4) From $s + 2 < T < s + 2 + 4/s$ again, we have $\lim_{s \rightarrow \infty} T/s = 1$, which implies $\lim_{s \rightarrow \infty} s/t = 1$ \square

Lemma 5.2. (1) $\lim_{s \rightarrow +0} B_s = 1$.

$$(2) \lim_{s \rightarrow \infty} B_s t^2 = 1.$$

Proof. (1) By Lemma 4.5,

$$B_s = -\frac{w_{2,1}}{w_{1,2}\sigma} = \frac{t - s - 1}{-1 + (1 + s)t}.$$

Lemma 5.1(1) implies $\lim_{s \rightarrow +0} B_s = 1$.

(2) We decompose $B_s t^2$ as

$$B_s t^2 = (t - s - 1) \cdot \frac{t^2}{-1 + (1 + s)t}.$$

From Lemma 5.1(3) and (4),

$$\lim_{s \rightarrow \infty} (t - s - 1) = \lim_{s \rightarrow \infty} \frac{t^2}{-1 + (1 + s)t} = 1.$$

Hence $\lim_{s \rightarrow \infty} B_s t^2 = 1$. \square

Proposition 5.3. The image of g contains an open interval $(0, 4)$.

Proof. By Lemma 5.2(1), $\lim_{s \rightarrow +0} \log B_s = 0$. Hence

$$\lim_{s \rightarrow +0} g(s) = -\lim_{s \rightarrow +0} \frac{\log B_s}{\log A_s} = -\lim_{s \rightarrow +0} \frac{\log B_s}{\log \sqrt{t}} = 0.$$

Also, we have $\lim_{s \rightarrow \infty} (\log B_s + 2 \log t) = 0$ by Lemma 5.2(2). Thus

$$\lim_{s \rightarrow \infty} g(s) = -\lim_{s \rightarrow \infty} \frac{\log B_s}{\log A_s} = -\lim_{s \rightarrow \infty} \frac{2 \log B_s}{\log t} = 4.$$

Hence the image of g contains an interval $(0, 4)$. \square

A computer experiment suggests that the image of g equals to $(0, 4)$, but we do not need this.

6. UNIVERSAL COVERING GROUP

We briefly review the description of the universal covering group of $SL_2(\mathbb{R})$.

Let

$$SU(1, 1) = \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \mid |\alpha|^2 - |\beta|^2 = 1 \right\}$$

be the special unitary group over \mathbb{C} of signature $(1, 1)$. It is well known that $SU(1, 1)$ is conjugate to $SL_2(\mathbb{R})$ in $GL_2(\mathbb{C})$. The correspondence is given by $\psi : SL_2(\mathbb{R}) \rightarrow SU(1, 1)$, sending $A \mapsto JAJ^{-1}$, where

$$J = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}.$$

Thus

$$\psi : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \frac{a+d+(b-c)i}{2} & \frac{a-d-(b+c)i}{2} \\ \frac{a-d+(b+c)i}{2} & \frac{a+d-(b-c)i}{2} \end{pmatrix}.$$

There is a parametrization of $SU(1, 1)$ by (γ, ω) where $\gamma = \beta/\alpha$ and $\omega = \arg \alpha$ defined mod 2π (see [1]). Thus $SU(1, 1) = \{(\gamma, \omega) \mid |\gamma| < 1, -\pi \leq \omega < \pi\}$. Topologically, $SU(1, 1)$ is an open solid torus $\Delta \times S^1$, where $\Delta = \{\gamma \in \mathbb{C} \mid |\gamma| < 1\}$. The group operation is given by $(\gamma, \omega)(\gamma', \omega') = (\gamma'', \omega'')$, where

$$(6.1) \quad \gamma'' = \frac{\gamma' + \gamma e^{-2i\omega'}}{1 + \gamma\bar{\gamma}'e^{-2i\omega'}},$$

$$(6.2) \quad \omega'' = \omega + \omega' + \frac{1}{2i} \log \frac{1 + \gamma\bar{\gamma}'e^{-2i\omega'}}{1 + \bar{\gamma}\gamma'e^{2i\omega'}}.$$

These equations come from the matrix operation. Here, the logarithm function is defined by its principal value and ω'' is defined by mod 2π . The identity element is $(0, 0)$, and the correspondence between $\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}$ and (γ, ω) gives an isomorphism.

Now, the universal covering group $\widetilde{SL_2(\mathbb{R})}$ of $SU(1, 1)$ can be described as

$$\widetilde{SL_2(\mathbb{R})} = \{(\gamma, \omega) \mid |\gamma| < 1, -\infty < \omega < \infty\}.$$

Thus $\widetilde{SL_2(\mathbb{R})}$ is homeomorphic to $\Delta \times \mathbb{R}$. The group operation is given by (6.1) and (6.2) again, but ω'' is not mod 2π anymore.

Let $\chi : \widetilde{SL_2(\mathbb{R})} \rightarrow SL_2(\mathbb{R})$ be the covering projection. Then $\ker \chi = \{(0, 2m\pi) \mid m \in \mathbb{Z}\}$.

Lemma 6.1. *The subset $(-1, 1) \times \{0\}$ of $\widetilde{SL_2(\mathbb{R})}$ forms a subgroup.*

Proof. From (6.1) and (6.2), it is straightforward to see that $(-1, 1) \times \{0\}$ is closed under the group operation. For $(\gamma, 0) \in (-1, 1) \times \{0\}$, its inverse is $(-\gamma, 0)$. \square

For the representation $\rho_s : G \rightarrow SL_2(\mathbb{R})$ defined by (2.1),

$$(6.3) \quad \psi(\rho_s(x)) = \frac{1}{2\sqrt{t}} \begin{pmatrix} t+1 & t-1 \\ t-1 & t+1 \end{pmatrix} \in SU(1, 1).$$

Thus $\psi(\rho_s(x))$ corresponds to $(\gamma_x, 0)$, where $\gamma_x = (t-1)/(t+1)$. Since $t > 1$, $\gamma_x \in (-1, 1)$.

Also, for the longitude \mathcal{L} ,

$$\psi(\rho_s(\mathcal{L})) = \frac{1}{2} \begin{pmatrix} B_s + \frac{1}{B_s} & B_s - \frac{1}{B_s} \\ B_s - \frac{1}{B_s} & B_s + \frac{1}{B_s} \end{pmatrix}, B_s > 0$$

from Proposition 4.2. Thus $\psi(\rho_s(\mathcal{L}))$ corresponds to $(\gamma_{\mathcal{L}}, 0)$, where $\gamma_{\mathcal{L}} = (B_s^2 - 1)/(B_s^2 + 1)$. Clearly, $\gamma_{\mathcal{L}} \in (-1, 1)$.

7. PROOF OF THEOREM 1.1

Since the knot exterior M_n of K_n satisfies $H^2(M_n; \mathbb{Z}) = 0$, any $\rho_s : G \rightarrow SL_2(\mathbb{R})$ lifts to a representation $\tilde{\rho} : G \rightarrow \widetilde{SL_2(\mathbb{R})}$ ([10]). Moreover, any two lifts $\tilde{\rho}$ and $\tilde{\rho}'$ are related as follows:

$$\tilde{\rho}'(g) = h(g)\tilde{\rho}(g),$$

where $h : G \rightarrow \ker \chi \subset \widetilde{SL_2(\mathbb{R})}$. Since $\ker \chi = \{(0, 2m\pi) \mid m \in \mathbb{Z}\}$ is isomorphic to \mathbb{Z} , the homomorphism h factors through $H_1(M_n)$, so it is determined only by the value $h(x)$ of a meridian x (see [15]).

The following result is the key in [3], which is originally claimed in [15], for the figure eight knot. Our proof most follows that of [3], but it is much simpler, because the values of $\psi(\rho_s(x))$ and $\psi(\rho_s(\mathcal{L}))$ are calculated explicitly in Section 6.

Lemma 7.1. *Let $\tilde{\rho} : G \rightarrow \widetilde{SL_2(\mathbb{R})}$ be a lift of ρ_s . Then replacing $\tilde{\rho}$ by a representation $\tilde{\rho}' = h \cdot \tilde{\rho}$ for some $h : G \rightarrow \widetilde{SL_2(\mathbb{R})}$, we can suppose that $\tilde{\rho}(\pi_1(\partial M_n))$ is contained in the subgroup $(-1, 1) \times \{0\}$ of $SL_2(\mathbb{R})$.*

Proof. Since $\chi(\tilde{\rho}(\mathcal{L})) = (\gamma_{\mathcal{L}}, 0)$, $\tilde{\rho}(\mathcal{L}) = (\gamma_{\mathcal{L}}, 2j\pi)$ for some j . On the other hand, \mathcal{L} is a commutator, because our knot is genus one. Therefore the inequality (5.5) of [23] implies $-3\pi/2 < 2j\pi < 3\pi/2$. Thus we have $\tilde{\rho}(\mathcal{L}) = (\gamma_{\mathcal{L}}, 0)$.

Similarly, $\tilde{\rho}(x) = (\gamma_x, 2l\pi)$ for some l . Let us choose $h : G \rightarrow \widetilde{SL_2(\mathbb{R})}$ so that $h(x) = (0, -2l\pi)$. Set $\tilde{\rho}' = h \cdot \tilde{\rho}$. Then a direct calculation shows that $\tilde{\rho}'(x) = (\gamma_x, 0)$ and $\tilde{\rho}'(\mathcal{L}) = (\gamma_{\mathcal{L}}, 0)$. Since x and \mathcal{L} generate the peripheral subgroup $\pi_1(\partial M_n)$, the conclusion follows from these. \square

Proof of Theorem 1.1. For $r = 0$, $M_n(0)$ is irreducible and has positive betti number. Hence $\pi_1(M_n(0))$ is left-orderable by [4, Corollary 3.4]. For $r = 4$, [6] and [22] confirmed the conclusion.

Let $r = p/q \in (0, 4)$. By Proposition 5.3, we can fix s so that $g(s) = r$. Choose a lift $\tilde{\rho}$ of ρ_s so that $\tilde{\rho}(\pi_1(\partial M_n)) \subset (-1, 1) \times \{0\}$. Then $\rho_s(x^p \mathcal{L}^q) = I$, so $\chi(\tilde{\rho}(x^p \mathcal{L}^q)) = I$. This means that $\tilde{\rho}(x^p \mathcal{L}^q)$ lies in $\ker \chi = \{(0, 2m\pi) \mid m \in \mathbb{Z}\}$. Hence $\tilde{\rho}(x^p \mathcal{L}^q) = (0, 0)$. Then $\tilde{\rho}$ can induce a homomorphism $\pi_1(M_n(r)) \rightarrow \widetilde{SL_2(\mathbb{R})}$ with non-abelian image. Recall that $\widetilde{SL_2(\mathbb{R})}$ is left-orderable ([2]). Hence any (non-trivial) subgroup of $SL_2(\mathbb{R})$ is left-orderable. Since $M_n(r)$ is irreducible ([12]), $\pi_1(M_n(r))$ is left-orderable by [4, Theorem 1.1]. This completes the proof. \square

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